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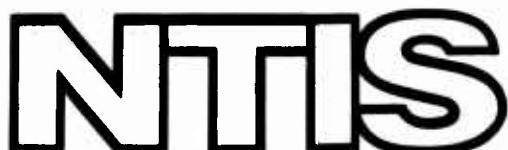
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	ROLE	WT	ROLE	WT	ROLE	WT
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ABSTRACT

A multicomponent reliability system in which each component is either up (i.e., working) or down (i.e, failed) in accordance with an alternating renewal process is considered. For arbitrary structures the following quantities are derived.

  (i) The average rate of system failure.

  (ii) The average uptime of the system.

  (iii) The average downtime of the system.

Further results are also obtained in the special case where the system structure is either series or parallel.

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MULTICOMPONENT RELIABILITY SYSTEMS

by

Sheldon M. Ross

1. INTRODUCTION

Consider an n-component reliability system having the property that at any time each of its components is either up (i.e., working) or down (i.e., being repaired). Let

$$X_i(t) = \begin{cases} 1 & \text{if } i^{th} \text{ component is up at time } t \\ 0 & \text{otherwise} \end{cases}$$

We suppose that $\{X_i(t), t \geq 0\}$, $i = 1, \ldots, n$, are independent alternating renewal processes. That is, for each i, there exist random variables U_j^i, D_j^i, $j \geq 1$, such that

$$X_i(t) = \begin{cases} 1 & \text{if } t \leq U_1^i \\ 0 & \text{if } U_1^i < t \leq U_1^i + D_1^i \\ 1 & \text{if } U_1^i + D_1^i \leq t < U_1^i + D_1^i + U_2^i \\ 0 & \text{etc.} \end{cases}$$

The random vectors $\left(U_j^i, D_j^i\right)$, $j \geq 1$, are assumed to be independent and identically distributed. Let

$$F_i(t) = P\left\{U_j^i \leq t\right\}$$

$$G_i(t) = P\left\{D_j^i \leq t\right\}$$

$$\mu_i = \int_0^\infty t \, dF_i(t)$$

$$v_i = \int_0^\infty t \, dG_i(t)$$

and suppose that $0 < \mu_i < \infty$, $0 < v_i < \infty$, $F_i(0) = G_i(0) = 0$.

We suppose that whether or not the total system is working at time t depends only on $\underline{X}(t) = (X_1(t), \ldots, X_n(t))$. In particular, letting

$$X(t) = \begin{cases} 1 & \text{if the system is working at time } t \\ 0 & \text{otherwise} \end{cases}$$

we assume that there exists a nondecreasing binary function ϕ such that

$$X(t) = \phi(X_1(t), \ldots, X_n(t))$$

It follows from the independence of components that $P\{X(t) = 1\} = E[\phi(\underline{X}(t))]$ is a function of $E[X_1(t)], \ldots, E[X_n(t)]$ and we write

$$P\{X(t) = 1\} = h(E[X_1(t)], \ldots, E[X_n(t)])$$

The function h is called the reliability function of the system. In words, $h(p_1, \ldots, p_n)$ represents the probability that the system will function when the ith component (independently of other components) functions with probability p_i . Assuming (as we do throughout) that the distribution of $U_1^i + D_1^i$ is not lattice, $i = 1, \ldots, n$, it follows from known alternating renewal process results that

$$E[X_i(t)] \rightarrow \frac{\mu_i}{\mu_i + v_i} \quad \text{as} \quad t \rightarrow \infty$$

and thus, from the continuity of h (which is easily shown), we obtain that

$$(1.1) \qquad P\{X(t) = 1\} \rightarrow h\left(\frac{\mu_1}{\mu_1 + v_1}, \ldots, \frac{\mu_n}{\mu_n + v_n}\right) \equiv h\left(\frac{\underline{\mu}}{\underline{\mu} + \underline{v}}\right)$$

a result first noted by Esary and Proschan [3]. The function $P\{X(t) = 1\}$ is referred to in the literature as the availability at time t , and thus Equation (1.1) states that the limiting availability is $h\left(\frac{\underline{\mu}}{\underline{\mu} + \underline{v}}\right)$.

2. REGENERATION POINTS

One unpleasant feature of the reliability process defined in Section 1 is that it does not necessarily have any regeneration points. That is, there need not be any time points at which the process "probabilistically starts over again." This is unfortunate as the theory of regenerative processes and the associated theory of renewal reward (or cumulative) processes yield elegant results concerning not only the existence of limits, but also the equality of different types of limits such as almost sure limits and limits in expectation. However, there are certain special cases for which the process is regenerative. One of these being if for each $i = 1, \ldots, n$, either F_i or G_i is a mixture of an exponential and some other distribution. That is, if for each i , either $F_i(t)$ or $G_i(t)$ is of the form

$$(21) \qquad p_i\left(1 - e^{-\lambda_i t}\right) + (1 - p_i)H_i(t)$$

where $0 < p_i < 1$, and H_i is a distribution function, then the process is regenerative with a finite mean regeneration cycle[†]; the regeneration times being those times at which the "exponential parts" of each component are in effect. In fact since the p_i in Equation (2.1) can be arbitrarily small (though positive) it follows that any set of distributions $\{F_i, G_i, i = 1, \ldots, n\}$ can be approximated arbitrarily closely by one of the forms (2.1). As a result, we shall assume that our reliability process is a regenerative process with finite mean regeneration cycle. In fact since our results will only depend on μ_i and v_i , $i = 1, \ldots, n$, and as any F_i (or G_i) can be arbitrarily closely approximated by a distribution of the form (2.1) it is intuitively obvious that our results hold in the general case. (In fact, it would seem that appropriate applications of the ergodic theorem yields the same existence and equality of limit results as does the assumption of regeneration points; thus proving our results in the general case.)

[†] The finite mean regeneration time follows from the result that a finite state Markov chain has no null recurrent states.

3. RATE OF SYSTEM BREAKDOWN

We say that component i causes a breakdown at time t if

(i) $X_i(t^-) = 1$

(ii) $X_i(t) = 0$

(iii) $X(t^-) = 1$

(iv) $X(t) = 0$

Let $N_i(t)$ denote the number of breakdowns caused by i in $[0,t]$.

Notation:

Let $\underline{P} = (P_1, \ldots, P_n)$. Define

$$(1_i, \underline{P}) = (P_1, \ldots, P_{i-1}, 1, P_{i+1}, \ldots, P_n)$$

$$(0_i, \underline{P}) = (P_1, \ldots, P_{i-1}, 0, P_{i+1}, \ldots, P_n)$$

Proposition 3.1:

(a) With probability 1,

$$\lim_{t \to \infty} \frac{N_i(t)}{t} = \lim_{t \to \infty} \frac{E[N_i(t)]}{t} = \frac{h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)}{\mu_i + v_i}$$

(b) $$E[N_i(t + a) - N_i(t)] \to \frac{a\left[h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)\right]}{\mu_i + v_i}$$

Proof:

From the theory of renewal reward processes it follows that, with probability 1,

$$\lim \frac{N_i(t)}{t} = \lim \frac{E[N_i(t)]}{t}$$

$$= E[\text{Number of breakdowns caused by } i \text{ in a regeneration cycle}]/E[\text{Time of a regeneration cycle}]$$

Define

$$
I_j = \begin{cases} 1 & \text{if the } j^{th} \text{ breakdown of } i \text{ causes a system breakdown} \\ 0 & \text{otherwise} \end{cases}
$$

Now, again by the theory of renewal reward processes it follows that, with probability 1,

$$
(3.1) \qquad \lim_k \frac{I_1 + \ldots + I_k}{k} = \lim_k \frac{E[I_1 + \ldots + I_k]}{k}
$$

$$
= E[\text{Number of breakdowns caused by } i \text{ in a regeneration cycle}]/E[\text{Number of breakdowns of } i \text{ in a regeneration cycle}]
$$

However, from the independence of components, it follows that

$$
(3.2) \qquad E(I_j) \to h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right) \quad \text{as} \quad j \to \infty
$$

This is true since the jth failure of i will cause a system failure if the state vector $\underline{x}$ at that time is such that $\phi(1_i, \underline{x}) = 1$, $\phi(0_i, \underline{x}) = 0$. We then obtain (3.2) since $P\{\phi(1_i, \underline{X}(t)) = 1, \phi(0_i, \underline{X}(t)) = 0\} = P\{\phi(1_i, \underline{X}(t)) = 1\} - P\{\phi(0_i, \underline{X}(t)) = 1\}$. Thus from (3.1) and (3.2) we see that, with probability,

$$
\frac{I_1 + \ldots + I_k}{k} \to h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)
$$

Now, since $\lim \frac{N_i(t)}{t}$ exists with probability 1, it follows that

$$
\lim_{t \to \infty} \frac{N_i(t)}{t} = \lim_{k \to \infty} \frac{N_i(\text{time of } k^{th} \text{ breakdown of } i)}{\text{time of } k^{th} \text{ breakdown of } i}
$$

$$
= \lim_k \frac{I_1 + \ldots + I_k}{k} \frac{k}{\text{time of } k^{th} \text{ breakdown of } i}
$$

$$= \frac{h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)}{\mu_i + v_i}$$

Part (b) follows from (a) since Blackwell's Theorem (of renewal theory) holds for renewal reward processes (see [ ]). ||

Letting $N(t)$ denote the number of breakdowns in $(0, t)$, we obtain, from Proposition 3.1

Corollary 3.2:

(a) With probability 1,

$$\lim_{t \to \infty} \frac{N(t)}{t} = \lim \frac{E[N(t)]}{t} = \sum_{i=1}^{n} (\mu_i + v_i)^{-1} \left[h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)\right]$$

(b) $E[N(t + a) - N(t)] \to a \sum_{i=1}^{n} (\mu_i + v_i)^{-1} \left[h\left(1_i, \frac{\mu}{\mu + v}\right) - h\left(0_i, \frac{\mu}{\mu + v}\right)\right]$

Proof:

(a) The nonlattice assumption implies that $E[N(t)] = E\left[\sum_{i=1}^{n} N_i(t)\right] + o(t)$ and so (a) follows. (It should be noted that $N(t) \neq \sum_{1}^{n} N_i(t)$ since more than 1 component can be given credit for a breakdown at time t.)

(b) Follows from (a) since Blackwell's Theorem holds for renewal reward processes. ||

We say that component i causes an uptime at time t if

   (i) $X_i(t^-) = 0$

  (ii) $X_i(t) = 1$

 (iii) $X(t^-) = 0$

 (iv) $X(t) = 1$

Let $N_i^*(t)$ denote the number of uptimes caused by i in $[0,t]$. By a proof completely analagous to that of Proposition 3.1 we can prove

Proposition 3.3:

(a) With probability 1,

$$\lim_t \frac{N_i^*(t)}{t} = \lim_t \frac{E[N_i^*(t)]}{t} = \frac{h\left(1_i, \frac{\mu}{\mu+v}\right) - h\left(0_i, \frac{\mu}{\mu+v}\right)}{\mu_i + v_i}$$

(b) $$E\left[N_i^*(t + a) - N_i^*(t)\right] \rightarrow \frac{a\left[h\left(1_i, \frac{\mu}{\mu+v}\right) - h\left(0_i, \frac{\mu}{\mu+v}\right)\right]}{\mu_i + v_i}$$

Remark:

It is interesting to note that the proportion of downtimes (i.e., breakdowns) caused by component i is equal to the proportion of uptimes that it causes.

4. UPTIMES AND DOWNTIMES

Let $U(t)$ denote the cumulative amount of uptime in $[0,t]$. From the theory of renewal reward processes it follows that, with probability 1,

$$\frac{U(t)}{t} - P\{X(t) = 1\} \to 0 \quad \text{as} \quad t \to \infty$$

and thus, from Equation (1.1),

(4.1) $\qquad \frac{U(t)}{t} \to h\left(\frac{\mu}{\mu + v}\right) \quad$ as $\quad t \to \infty \quad$ (with probability 1).

The reliability system will alternate between periods in which the system is up and periods in which it is down. Let us denote by U_i the length of the ith up period, $i = 1$, and by D_i, the length of the ith down period. It should be noted that the U_i, $i \geq 1$ (and similarly, the D_i) are neither independent nor are they identically distributed random variables. However it does follow from results presented in [2] that

$$\lim_{n \to \infty} \frac{U_1 + \ldots + U_n}{n} = \lim E[U_1 + \ldots + U_n]/n = E[U_\infty]$$

and

$$\lim_{n \to \infty} \frac{D_1 + \ldots + D_n}{n} = \lim E[D_1 + \ldots + D_n]/n = E[D_\infty]$$

where U_∞ (D_∞) is a random variable whose distribution is the limiting distribution of U_n (D_n).

Moreover from Equation (4.1) we note that

$$\frac{\sum_{i=1}^{n} U_i}{\sum_{1}^{n} U_i + \sum_{1}^{n} D_i} \to h\left(\frac{\mu}{\mu + v}\right) \quad \text{as} \quad n \to \infty$$

or, equivalently

$$\frac{E[U_\infty]}{E[U_\infty] + E[D_\infty]} = h\left(\frac{\mu}{\mu + v}\right)$$

Finally, as $\lim\limits_{n} \dfrac{n}{\sum\limits_{i=1}^{n} (U_i + D_i)} = \dfrac{1}{E[U_\infty] + E[D_\infty]}$ represents the average rate at

which breakdowns occur, we obtain from Corollary 3.2

Proposition 4.1:

$$E[U_\infty] = \frac{h\left(\frac{\mu}{\mu+v}\right)}{\sum\limits_{i=1}^{n} (\mu_i + v_i)^{-1}\left[h\left(1_i, \frac{\mu}{\mu+v}\right) - h\left(0_i, \frac{\mu}{\mu+v}\right)\right]}$$

$$E(D_\infty) = \frac{1 - h\left(\frac{\mu}{\mu+v}\right)}{\sum\limits_{i=1}^{n} (\mu_i + v_i)^{-1}\left[h\left(1_i, \frac{\mu}{\mu+v}\right) - h\left(0_i, \frac{\mu}{\mu+v}\right)\right]}$$

5. SPECIAL CASES

5.1 Series Structure

The reliability system is said to be a series system if in order for it to be up it is necessary that all of the components be up. Hence, for a series system

$$h(\underline{P}) = \prod_{i=1}^{n} P_i$$

Hence, from Corollary 3.2, we obtain that the rate of breakdowns of a series system is given by

$$(5.1) \qquad \lim \frac{N(t)}{t} = \prod_{j=1}^{n} \frac{\mu_j}{\mu_j + v_j} \sum_{i=1}^{n} 1/\mu_i$$

while the average length of up and down periods is

$$E[U_\infty] = \frac{1}{\sum\limits_{i=1}^{n} 1/\mu_i}$$

(5.2)

$$E[D_\infty] = \frac{1 - \prod\limits_{i=1}^{n} \frac{\mu_i}{\mu_i + v_i}}{\prod\limits_{j=1}^{n} \frac{\mu_j}{\mu_j + v_j} \sum\limits_{i=1}^{n} 1/\mu_i}$$

Remark:

If all component uptime distributions were exponential then $E[U_n] = \left[\sum\limits_{1}^{n} 1/\mu_i \right]^{-1}$ for all n . It is interesting to note that this is also the average of system uptimes for arbitrary distributions. (This result was also obtained in [1] where the model considered was series but with the provision that when the system failed the remaining components did not age but were held in a state of suspended

animation until the failed component was repaired.) Furthermore, when the uptime distributions are exponential then $D_1, D_2, \ldots$ are independent and identically distributed and thus

$$E[D_n] = E[D_\infty] = \frac{1 - \prod_{i=1}^{n} \frac{\mu_i}{\mu_i + v_i}}{\prod_{j=1}^{n} \frac{\mu_j}{\mu_j + v_j} \sum_{i=1}^{n} 1/\mu_i} \quad \text{for all } n.$$

It is interesting to note that this result (in the exponential component uptimes case) seems to be difficult to prove directly.||

Additional insight into the length of uptime periods is obtained by the following heuristic reasoning. Given that (the fixing of) component i has just caused an uptime period it is intuitive that the remaining uptime of component j, $j \neq i$, should be given by the equilibrium distribution $F_{j,e}(x)$, where

$$F_{j,e}(x) = \frac{\int_0^x (1 - F_j(x)) dx}{\mu_j}$$

Hence, as the proportion of uptimes that are caused by component i equals

$$\lim_{t \to \infty} \frac{N_i^*(t)}{\sum_{j=1}^{n} N_j^*(t)} = \frac{\frac{1}{\mu_i}}{\sum_{1}^{n} 1/\mu_j}, \quad \text{it follows upon conditioning that}$$

$$(5.3) \qquad P\{U_\infty > x\} = \left(\sum_{j=1}^{n} 1/\mu_j \right)^{-1} \sum_{i=1}^{n} \frac{1}{\mu_i}(1 - F_i(x)) \prod_{j \neq i} (1 - F_{j,e}(x))$$

represents the limiting distribution of U_n. Another interpretation of (5.3) is that it represents the proportion of uptime periods that are of length greater than x. If Equation (5.3) is indeed valid then, as $E[U_\infty] = \int_0^\infty P\{U_\infty > x\} dx$,

it would be necessary for $\displaystyle\int_0^\infty \sum_{i=1}^n \frac{1}{\mu_i}(1 - F_i(x)) \prod_{j\neq i}(1 - F_{j,e}(x))dx$ to equal 1.
We now verify this.

Proposition 5.1:

$$\int_0^\infty \sum_{i=1}^n \frac{1 - F_i(x)}{\mu_i} \prod_{j\neq i}(1 - F_{j,e}(x))dx = 1 .$$

Proof:

The above appears difficult to verify analytically, but there is a simple

probabilistic proof. Noting that $\dfrac{1 - F_i(x)}{\mu_i}dx = dF_{i,e}(x)$ we see that the above

integral is equal to

$$\sum_{i=1}^n \int_0^\infty \prod_{j\neq i}(1 - F_{j,e}(x))dF_{i,e}(x) .$$

Letting $X_1, \ldots, X_n$ be independent random variables, with X_i having distri-
bution $F_{i,e}(x)$ we note by a simple conditioning argument that

$$P\{X_i = \text{smallest of } (X_1, \ldots, X_n)\} = \int_0^\infty \prod_{j\neq i}(1 - F_{j,e}(x))dF_{i,e}(x)$$

and thus, summing over all i fields the desired result.$\|$

Remark:

It also follows from Equation (5.3) that if each F_i has a DFR distribu-
tion; that is, if $\dfrac{1 - F(t + s)}{1 - F(t)} \uparrow t$ for all s , then the distribution of U_∞
is also DFR. This follows from the 3 facts

(1) Mixtures of DFR distributions are themselves DFR.

(ii) If F is DFR then F_e is also DFR (this follows from (i) using
results about the equilibrium renewal process, it can also be
shown directly).

(iii) A series system made up of independent DFR component lifetimes, has
a DFR lifetime distribution.

In fact the same argument shows that U_n is DFR for all n .

Another quantity of interest is the remaining uptime of a system that is
up at time t (t large). Again, heuristic reasoning yields that

$$(5.4) \qquad P\{\text{additional uptime} > x\} = \prod_{j=1}^{n} (1 - F_{j,e}(x))$$

and thus

$$E[\text{additional uptime}] = \int_0^{\infty} \prod_{j=1}^{n} (1 - F_{j,e}(x))dx$$

$$= \mu_{1,e} \int_0^{\infty} \frac{1 - F_{1,e}(x)}{\mu_{1,e}} \prod_{j\neq 1} (1 - F_{j,e}(x))dx$$

$$= \mu_{1,e} \int_0^{\infty} \prod_{j\neq 1} (1 - F_{j,e}(x))dF_{1,e,e}(x)$$

where $\mu_{1,e} = \int_0^{\infty} xdF_{1,e}(x) = \int_0^{\infty} \frac{x^2 dF_1(x)}{2\mu_1}$. Hence,

$$E[\text{additional uptime}] = \mu_{1,e} P\{X_{1,e,e} = \text{smallest of } (X_{1,e,e}, X_{2,e}, \ldots, X_{n,e})\}$$

where $X_{1,e,e}$, $X_{1,e}$, $i \geq 2$, are independent and $X_{1,e,e} \sim F_{1,e,e}$ and
$X_{1,e} \sim F_{1,e}$, $i \geq 2$. In the special case where all the component uptime

distributions are identical and the distribution is IFR $\Big($ that is,

$\dfrac{1 - F(t + s)}{1 - F(t)} \downarrow t$ for all $s \Big)$ it follows that the expected additional system

uptime of a series system that is up at time infinity satisfies the following

inequality

$$E[\text{additional uptime}] \ge \frac{\mu_{1,e}}{n} = \frac{\displaystyle\int_0^\infty x^2 dF(x)}{2n\mu} .$$

This inequality is reversed if F is DFR. The above inequality follows from

the fact (proven below) that if F is IFR then so is F_e .

It again follows that if each F_i is DFR then the distribution of addi-

tional life as given by Equation (5.4) is also DFR. Moreover if each F_i is

IFR then it follows that the additional life distribution is also IFR. This

follows from the fact that a series system of independent IFR components itself

has an IFR life distribution upon application of the following proposition.

Proposition 5.2:

If F is an IFR distribution with finite mean, then F_e is also IFR where

$$F_e(t) = \int_0^t (1 - F(y))dy / \int_0^\infty (1 - F(y))dy$$

Proof:

The failure rate function of F_e is given by

$$\lambda_e(t) = \left[\frac{1 - F_e(t)}{\dfrac{d}{dt} F_e(t)} \right]^{-1}$$

$$= \left[\int_t^\infty \frac{1 - F(y)}{1 - F(t)} \, dy \right]^{-1}$$

$$= \left[\int_0^\infty \frac{1 - F(t + x)}{1 - F(t)} \, dx \right]^{-1}$$

$= 1/E[\text{additional life of a } t \text{ year old item having life distribution } F]$

and the result follows since F IFR implies that the denominator of the above term is decreasing in t .||

It should be noted that it has been shown in [4] that if all component uptime and downtime distributions are exponential then both the limiting distribution of system uptime U_n and of the additional uptime of a system that is up are both mixtures of exponential distributions and are thus DFR. These results hold for Markovian (i.e., exponential) systems having an arbitrary structure (that is, they need not be series).

5.2 Parallel Structure

The reliability system is said to be a parallel system if it is up whenever at least one of its components is up. That is,

$$h(\underline{P}) = 1 - \prod_1^n (1 - P_i) \ .$$

In other words it is down if and only if all of its components are down. Thus we see that by regarding down as up the parallel system is transformed into a series system and so all of the results of Section 5.1 are immediately translatable into results about parallel systems. For example, from Equations (5.1) and (5.2) we see that, in the parallel case,

$$E[D] = \frac{1}{\sum 1/v_i}$$

$$E[U_\infty] = \frac{1 - \prod_{i=1}^n \frac{v_i}{\mu_i + v_i}}{\prod_{n=1}^n \frac{v_i}{\mu_j + v_j} \sum_{i=1}^n 1/v_i}$$

$$\lim \frac{N(t)}{t} = \prod_{j=1}^{n} \frac{v_j}{\mu_j + v_j} \sum_{j=1}^{n} 1/v_i \ .$$

5.3 The k-out-of-n Structure with Identical Component Distributions

Series and parallel systems are both special cases of the k-out-of-n system which is up if at least k of the n components are up. Let us consider such a system along with the additional assumption that $\mu_i \equiv \mu$, and $v_i \equiv v$, $i = 1, \ldots, n$. For this case we obtain that

$$h\left(1_i, \frac{\mu}{\mu+v}\right) - h\left(0_i, \frac{\mu}{\mu+v}\right)$$

$$= \sum_{i=k-1}^{n-1} \binom{n-1}{i}\left(\frac{\mu}{\mu + v}\right)^i\left(\frac{v}{\mu + v}\right)^{n-1-i} - \sum_{i=k}^{n-1} \binom{n-1}{i}\left(\frac{\mu}{\mu + v}\right)^i\left(\frac{v}{\mu + v}\right)^{n-1-i}$$

$$= \binom{n-1}{k-1}\left(\frac{\mu}{\mu + v}\right)^{k-1}\left(\frac{v}{\mu + v}\right)^{n-k}$$

Thus, from Corollary 3.2 we note that

$$\lim \frac{N(t)}{t} = \frac{n!}{(n - k)!(k - 1)!} \frac{\mu^{k-1}v^{n-k}}{(\mu + v)^n}$$

and from Proposition 4.1

$$E[U_\infty] = \frac{(n - k)!(k - 1)!}{n!\mu^{k-1}v^{n-k}} \sum_{i=k}^{n} \binom{n}{i}\mu^i v^{n-i}$$

$$E[D_\infty] = \frac{(n - k)!(k - 1)!}{n!\mu^{k-1}v^{n-k}} \sum_{i=0}^{k-1} \binom{n}{i}\mu^i v^{n-i}$$

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